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Flow in a rotating non-aligned straight pipe

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A third-order regular perturbation solution is developed for laminar flow through a straight pipe that is rotating about an axis not aligned with the pipe axis. Coriolis accelerations produce transverse secondary velocities (similar to those in flow through coiled tubes) and modify the axial-velocity profile. The effects of rotation on the velocity fields are shown to depend on two parameters: (i) the product of axial and rotational Reynolds numbers, and (ii) the square of the rotational Reynolds number itself. Even though their strength increases with increases in parameter magnitudes, transverse circulations are qualitatively insensitive to parametric values. The axial profile, on the other hand, can be significantly modified by the rotation; the zeroth-order parabolic axial profile can be skewed toward the outside, dimpled in the centre with maximums on either side of the centreline, or both, depending on the values of the two parameters. The modification of the axial-velocity profile has important ramifications in the design of heat/mass-transfer devices.

1. Introduction

Transverse laminar secondary motions in pipe flow has been the subject of numerous studies (e.g. Dean 1927, 1928; Barua 1954, 1963; Benton 1956; McConalogue & Srivastava 1968; Ito & Nanbu 1971; Ito & Motai 1974; Van Dyke 1978; Manlapoz & Churchill 1980; Nandakumar & Masliyah 1982). These secondary flows are of particular interest for their effect on heat and/or mass transfer to or from fluids flowing axially through the pipes. The laminar secondary motions result in only modest increases in axial pressure losses while producing large increases (often by an order of magnitude or more) in the rates of heat and/or mass transfer. The present paper presents a solution for the flow of a Newtonian fluid through a straight pipe that is rotating about an axis oriented at an angle α relative to the centreline of the pipe. The angular velocity Ω of the pipe is taken as positive when oriented in the α-direction and negative in the opposite direction (see figure 1). The rotation subjects the axially flowing fluid to centrifugal and Coriolis accelerations. The centrifugal acceleration only subjects the fluid to a pseudo body force and does not produce a flow disturbance. The Coriolis acceleration, however, produces transverse secondary circulations. The solution is a straightforward extension of the regular perturbation-series solution given by Barua (1954) for flow in a pipe rotating about an axis perpendicular to the pipe axis. Barua, however, was primarily interested in determining the friction-loss versus flow-rate relationship, and, as a result, developed the first-order and only a portion of the second-order solution. The present solution is carried out to third order and is shown to involve two parameters $N_R N_\alpha$ and N_α^2, in which $N_R \equiv$ axial Reynolds number $\equiv 2aW/\nu$ and $N_\alpha \equiv$ rotational Reynolds number

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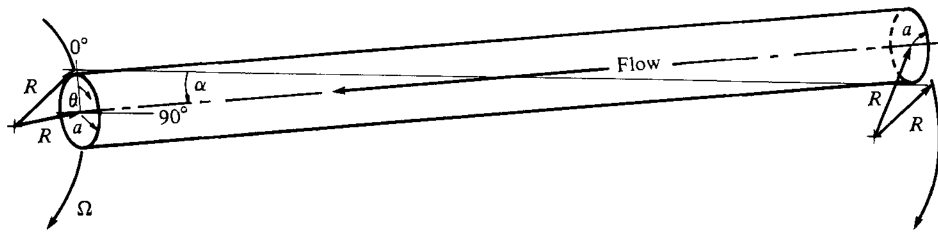

FIGURE 1. Sketch of a rotating non-aligned straight pipe.

$\equiv \Omega a^2 \sin\alpha/48\nu$, where $a \equiv$ pipe radius, $W \equiv$ cross-sectional average axial velocity and $\nu \equiv$ kinematic viscosity of the fluid. Besides inducing transverse secondary velocities, the rotation alters the axial-velocity profile. In the regime where N_α^2 is small but $N_R N_\alpha$ is not, the effect of rotation is to skew the paraboloid profile toward the outside, i.e. rotation moves the maximum out along $\theta = 90°$ in figure 1. In the regime where $N_R N_\alpha$ is small but N_α^2 is not, the effect of rotation is to reduce the centreline velocity and produce an axial-velocity profile with two maxima, one along $\theta = 0°$ and one along $\theta = 180°$. If both $N_R N_\alpha$ and N_α^2 are of significant size, the axial profile is skewed with two maxima. The first-order term of the series produces skewing toward the outside, the second-order term reduces the centreline velocity and produces double maxima, and the third-order term tends to negate the effect of the first-order term by producing skewing toward the inside.

2. Governing equations

The momentum equations may be conveniently expressed using circular cylindrical coordinates (r^*, θ, z^*) fixed to the rotating pipe (see figure 1):

$$u^*\frac{\partial u^*}{\partial r^*}+\frac{v^*}{r^*}\frac{\partial u^*}{\partial \theta}-\frac{v^{*2}}{r^*} = 2w^*\Omega\sin\theta\sin\alpha+2v^*\Omega\cos\alpha-\frac{1}{\rho}\frac{\partial p^*}{\partial r^*} +\nu\left[\nabla^{*2}u^*-\frac{u^*}{r^{*2}}-\frac{2}{r^{*2}}\frac{\partial v^*}{\partial\theta}\right], \quad (1)$$

$$u^*\frac{\partial v^*}{\partial r^*}+\frac{v^*}{r^*}\frac{\partial v^*}{\partial \theta}+\frac{u^*v^*}{r^*} = 2w^*\Omega\cos\theta\sin\alpha-2u^*\Omega\cos\alpha-\frac{1}{\rho r^*}\frac{\partial p^*}{\partial \theta} +\nu\left[\nabla^{*2}v^*-\frac{v^*}{r^{*2}}+\frac{2}{r^{*2}}\frac{\partial u^*}{\partial\theta}\right], \quad (2)$$

$$u^*\frac{\partial w^*}{\partial r^*}+\frac{v^*}{r^*}\frac{\partial w^*}{\partial \theta} = -2v^*\Omega\cos\theta\sin\alpha-2u^*\Omega\sin\theta\sin\alpha-\frac{1}{\rho}\frac{\partial p^*}{\partial z^*}+\nu[\nabla^{*2}w^*], \quad (3)$$

in which u^*, v^* and w^* are the velocity components in the directions of r^*, θ and z^* respectively, ρ and ν are the density and the kinematic viscosity of the fluid respectively, and ∇^{*2} is the Laplacian operator. The terms containing Ω are due to Coriolis acceleration. The reduced pressure p^* represents the difference between the actual pressure p and the centrifugal forces in the rotating system:

$$p^* = p-\tfrac{1}{2}\rho\Omega^2[(z^2+r^2\sin^2\theta)\sin^2\alpha+r^2\cos^2\alpha-2rz\cos\alpha\sin\alpha\cos\theta+2Rr\sin\theta+R^2]. \quad (4)$$

The equation of continuity may be written as

$$\frac{1}{r^*}\frac{\partial(r^*u^*)}{\partial r^*}+\frac{1}{r^*}\frac{\partial v^*}{\partial\theta} = 0. \quad (5)$$

The boundary conditions, no slip at the pipe wall, are

$$u^* = v^* = w^* = 0 \quad \text{at} \quad r^* = a, \tag{6}$$

in which a is the radius of the tube. Equation (3) suggests that p^* must be of the form $z^*g_1(r^*, \theta)+g_2(r^*,\theta)$. Equations (1) and (2) indicate that $g_1(r^*,\theta)$ must be a constant, say G^*. The axial pressure gradient can therefore be written as

$$-\frac{\partial p^*}{\partial z^*} = G^*, \tag{7}$$

in which G^* is positive when the direction of the flow is in the direction of increasing z^*.

A stream function f^* can be defined for fully developed secondary flow:

$$u^* = -\frac{1}{r^*}\frac{\partial f^*}{\partial \theta}, \quad v^* = \frac{\partial f^*}{\partial r^*}. \tag{8a, b}$$

Inserting (8a,b) into (1) and (2) and eliminating p^* yields

$$\frac{1}{r^*}\frac{\partial(f^*, \nabla^{*2}f^*]}{\partial[r^*,\theta]} + 2\Omega \sin\alpha \left[\cos\theta \frac{\partial w^*}{\partial r^*} - \frac{\sin\theta}{r^*}\frac{\partial w^*}{\partial \theta}\right] = \nu\nabla^{*4}f^*, \tag{9}$$

in which $\nabla^{*4} = \nabla^{*2}(\nabla^{*2})$ and $\partial[A,B]/\partial[x,z]$ indicates the Jacobian. If y^* is defined as $r^*\cos\theta$,

$$\frac{\partial}{\partial y^*} = \frac{\partial}{\partial(r^*\cos\theta)} = \frac{\partial r^*}{\partial y^*}\frac{\partial}{\partial r^*} + \frac{\partial\theta}{\partial y^*}\frac{\partial}{\partial\theta} = \cos\theta\frac{\partial}{\partial r^*} - \frac{\sin\theta}{r^*}\frac{\partial}{\partial\theta}, \tag{10}$$

then (9) can be written as

$$\frac{1}{r^*}\frac{\partial[f^*, \nabla^{*2}f^*]}{\partial[r^*,\theta]} - 2\Omega\sin\alpha\frac{\partial w^*}{\partial y^*} = \nu\nabla^{*4}f^*. \tag{11}$$

Inserting (7), (8) and (10) into (3) results in

$$\frac{1}{r^*}\frac{\partial[f^*, w^*]}{\partial[r^*,\theta]} + 2\Omega\sin\alpha\frac{\partial f^*}{\partial y^*} = \frac{G^*}{\rho} + \nu\nabla^{*2}w^*. \tag{12}$$

Equations (11) and (12) may be cast into non-dimensional form by introducing the following dimensionless variables:

$$f \equiv f^*/\nu, \quad w \equiv w^*a/\nu, \tag{13), (14}$$

$$r = r^*/a, \quad y = y^*/a. \tag{15a, b}$$

They may be then transformed into

$$\nabla^4 f = \frac{1}{r}\frac{\partial[f, \nabla^2 f]}{\partial[r,\theta]} - 96N_\alpha\frac{\partial w}{\partial y} \tag{16}$$

and

$$\nabla^2 w = \frac{1}{r}\frac{\partial[f, w]}{\partial[r,\theta]} + 96N_\alpha\frac{\partial f}{\partial y} - 4G \tag{17}$$

respectively, in which

$$G = \frac{G^*a^3}{4\rho\nu^2} \tag{18}$$

and

$$N_\alpha = \frac{\Omega a^2 \sin\alpha}{48\nu}; \tag{19}$$

G is the Kármán number and N_α is an angular Reynolds number. The boundary conditions (6) that must be satisfied by (16) and (17) are

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial \theta} = w = 0 \quad \text{at} \quad r = 1, \tag{20}$$

and since the velocity of the fluid must be finite over the cross-section of the pipe

$$\frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial r} \text{ and } w \quad \text{are bounded in} \quad 0 \leqslant r \leqslant 1 \quad \text{and} \quad -\pi \leqslant \theta \leqslant \pi. \tag{21}$$

Equations (16), (17), (20) and (21) govern the motion of an incompressible Newtonian fluid flowing steadily through a straight pipe rotating at a constant angular velocity about an axis that is aligned at an angle α with respect to the pipe axis. An exact solution to this complex system of equations for $N_\alpha \neq 0$ is not available; asymptotic techniques, however, provide a method for obtaining approximate solutions for some ranges of the variables.

The solution to (16) and (17) is the well-known Hagen–Poiseuille flow for the limiting case of $N_\alpha = 0$, i.e. for the case of no rotation or rotation of a straight pipe about a parallel axis ($\alpha = 0°$). Thus the appearance of N_α is what renders the governing equations intractable. The N_α parameter is an angular Reynolds number that represents the ratio of Coriolis to viscous or frictional forces. Realistic estimates of N_α are less than unity, and therefore it can be a small parameter under practical conditions. The following perturbation analysis results in the approximation of w and f valid in the entire pipe cross-section for small value of N_α. The analysis is based on using N_α as a perturbation parameter, and a solution is sought of the following form:

$$w = w_0(r; G) + N_\alpha w_1(r, \theta; G) + N_\alpha^2 w_2(r, \theta; G) + N_\alpha^3 w_3(r, \theta; G) + \dots \tag{22}$$

and

$$f = N_\alpha f_1(r, \theta; G) + N_\alpha^2 f_2(r, \theta; G) + N_\alpha^3 f_3(r, \theta; G) + \dots. \tag{23}$$

3. Solution

Substituting (22) and (23) into (16) and (17) and equating coefficients of powers of N_α results in a series of relations that enable the successive determination of $w_0, w_1, w_2, \dots$ and $f_1, f_2, f_3, \dots$. The results of these manipulations are presented without including the extensive intermediate calculations.

The zeroth-order solution, terms of order N_α^0. This represents the case of no rotation (or rotation of a straight pipe about a parallel axis, $\alpha = 0°$), and is simply the flow through a stationary straight tube. The governing equation is

$$\nabla^2 w_0 = -4G, \tag{24}$$

with

$$w_0 = 0 \quad \text{at} \quad r = 1. \tag{25}$$

The solution is readily found to be

$$w_0 = G(1 - r^2). \tag{26}$$

Equation (26) is the zeroth-order solution to the overall problem of flow in a rotating non-aligned straight pipe; this stationary solution contains no secondary-flow patterns.

The first correction, terms of order N_α^1. The first approximation to the stream function for the secondary flow is determined from the following differential equation:

$$\nabla^4 f_1 = -96\frac{\partial w_0}{\partial y}. \tag{27}$$

Note that the solution of (27) provides the first term of the stream function for flow in a rotating non-aligned straight pipe assuming that the corresponding stationary straight pipe solution is known.

The boundary conditions follow from (20) and (21):

$$\frac{\partial f_1}{\partial r} = \frac{\partial f_1}{\partial \theta} = 0 \quad \text{at} \quad r = 1 \tag{28}$$

and

$$\frac{1}{r}\frac{\partial f_1}{\partial \theta}, \frac{\partial f_1}{\partial r} \quad \text{must be bounded in} \quad 0 \leqslant r \leqslant 1 \quad \text{and} \quad -\pi \leqslant \theta \leqslant \pi. \tag{29}$$

The solution is found to be

$$f_1 = Gr(1-r^2)^2 \cos\theta. \tag{30}$$

The first correction to the primary velocity profile is the solution of

$$\nabla^2 w_1 = \frac{1}{r}\frac{\partial[f_1, w_0]}{\partial[r,\theta]}, \tag{31}$$

with

$$w_1 = 0 \quad \text{at} \quad r = 1 \tag{32}$$

and the condition that

$$w_1 \quad \text{must be bounded in} \quad 0 \leqslant r \leqslant 1 \quad \text{and} \quad -\pi \leqslant \theta \leqslant \pi. \tag{33}$$

Note that w_1 can be determined assuming that w_0 and f_1 are known. The solution of (31) and (32) is

$$w_1 = \frac{G^2 r}{24}(1-r^2)(3-3r^2+r^4)\sin\theta. \tag{34}$$

The second correction, terms of order N_α^2. The second approximation to the stream function for the secondary flow is found from the differential equation

$$\nabla^4 f_2 = \frac{1}{r}\frac{\partial[f_1, \nabla^2 f_1]}{\partial[r,\theta]} - 96\frac{\partial w_1}{\partial y}, \tag{35}$$

which must satisfy the boundary condition

$$\frac{\partial f_2}{\partial r} = \frac{\partial f_2}{\partial \theta} = 0 \quad \text{at} \quad r = 1 \tag{36}$$

and the condition that

$$\frac{1}{r}\frac{\partial f_2}{\partial \theta}, \frac{\partial f_2}{\partial r} \quad \text{must be bounded in} \quad 0 \leqslant r \leqslant 1 \quad \text{and} \quad -\pi \leqslant \theta \leqslant \pi. \tag{37}$$

This term is found to be

$$f_2 = \frac{G^2}{480}r^2(1-r^2)^2(17-2r^2-r^4)\sin 2\theta. \tag{38}$$

The second correction to the primary-velocity profile is determined by solving

$$\nabla^2 w_2 = \frac{1}{r}\frac{\partial[f_2, w_\theta]}{\partial[r,\theta]} + \frac{1}{r}\frac{\partial[f_1, w_1]}{\partial[r,\theta]} + 96\frac{\partial f_1}{\partial y}, \tag{39}$$

with

$$w_2 = 0 \quad \text{at} \quad r = 1 \tag{40}$$

and the condition that

$$w_2 \quad \text{must be bounded in} \quad 0 \leqslant r \leqslant 1 \quad \text{and} \quad -\pi \leqslant \theta \leqslant \pi. \tag{41}$$

The solution is

$$\begin{aligned} w_2 = &-\frac{G^3(1-r^2)^4}{5760}(37-32r^2+10r^4) - 8G(1-r^2)^3 \\ &-\frac{G^3r^2(1-r^2)}{201\,600}(923-1457r^2+958r^4-302r^6+48r^8)\cos 2\theta \\ &+2Gr^2(1-r^2)(5-3r^2)\cos 2\theta. \end{aligned} \tag{42}$$

The third correction, terms of order N_α^3. The third approximation to the stream function for the secondary flow is governed by the differential equation

$$\nabla^4 f_3 = \frac{1}{r}\frac{\partial[f_1, \nabla^2 f_2]}{\partial[r,\theta]} + \frac{1}{r}\frac{\partial[f_2, \nabla^2 f_1]}{\partial[r,\theta]} - 96\frac{\partial w_2}{\partial y}, \tag{43}$$

with

$$\frac{\partial f_3}{\partial r} = \frac{\partial f_3}{\partial \theta} = 0 \quad \text{at} \quad r = 1 \tag{44}$$

and the condition that

$$\frac{1}{r}\frac{\partial f_3}{\partial \theta}, \frac{\partial f_3}{\partial r} \quad \text{must be bounded in} \quad 0 \leqslant r \leqslant 1 \quad \text{and} \quad -\pi \leqslant \theta \leqslant \pi. \tag{45}$$

Since the governing equation is

$$\nabla^4 f_3 = h_1(r)\cos\theta + h_2(r)\cos 3\theta \tag{46}$$

the solution is expected to have the form

$$f_3(r,\theta) = g_1(r)\cos\theta + g_2(r)\cos 3\theta, \tag{47}$$

in which

$$g_1(r) = g_2(r) = 0 \quad \text{at} \quad r = 1, \tag{48}$$

$$\frac{\partial g_1}{\partial r} = \frac{\partial g_2}{\partial r} = 0 \quad \text{at} \quad r = 1 \tag{49}$$

and the conditions that

$$\frac{\partial g_1(r)}{\partial r}, \frac{\partial g_2(r)}{\partial r} \quad \text{must be bounded in} \quad 0 \leqslant r \leqslant 1 \quad \text{and} \quad -\pi \leqslant \theta \leqslant \pi \tag{50}$$

and

$$\frac{1}{r}g_1, \frac{1}{r}g_2 \quad \text{must be bounded in} \quad 0 \leqslant r \leqslant 1 \quad \text{and} \quad -\pi \leqslant \theta \leqslant \pi. \tag{51}$$

The explicit solution is

$$
\begin{aligned}
f_3 = &-\frac{G(1-r^2)^2\,r}{5}(+77-42r^2+9r^4)\cos\theta \\
&-\frac{G^3r(1-r^2)^2}{2822400}(+23973-22607r^2+15891r^4-6371r^6+1110r^8-33r^{10})\cos\theta \\
&-\frac{2Gr^3(1-r^2)^2\,(4-r^2)}{5}\cos 3\theta \\
&-\frac{G^3r^3(1-r^2)^2}{2419200}(+895+1214r^2-861r^4+4r^6+29r^8)\cos 3\theta. \qquad (52)
\end{aligned}
$$

The differential equation characterizing the third correction to the primary velocity profile is

$$\nabla^2 w_3 = \frac{1}{r}\frac{\partial[f_3, w_0]}{\partial[r,\theta]} + \frac{1}{r}\frac{\partial[f_2, w_1]}{\partial[r,\theta]} + \frac{1}{r}\frac{\partial[f_1, w_2]}{\partial[r,\theta]} + 96\frac{\partial f_2}{\partial y}, \qquad (53)$$

with

$$w_3 = 0 \quad \text{at} \quad r = 1 \qquad (54)$$

and the condition that

$$w_3 \quad \text{must be bounded in} \quad 0 \leqslant r \leqslant 1 \quad \text{and} \quad -\pi \leqslant \theta \leqslant \pi. \qquad (55)$$

Equation (53) is of the form

$$\nabla^2 w_3 = H_1(r)\sin\theta + H_2(r)\sin 3\theta, \qquad (56)$$

and the solution has the form

$$w_3 = G_1(r)\sin\theta + G_2(r)\sin 3\theta, \qquad (57)$$

with

$$G_1(r) = G_2(r) = 0 \quad \text{at} \quad r = 1 \qquad (58)$$

and the condition that

$$G_1(r), G_2(r) \quad \text{must be bounded in} \quad 0 \leqslant r \leqslant 1 \quad \text{and} \quad -\pi \leqslant \theta \leqslant \pi. \qquad (59)$$

The actual solution is found to be

$$
\begin{aligned}
w_3 = &-\frac{G^2r(1-r^2)}{150}(548-682r^2+343r^4-57r^6+3r^8)\sin\theta \\
&-\frac{G^4(1-r^2)\,r}{677376000}(2150487-6104913r^2+9716907r^4-9504533r^6 \\
&+5877547r^8-2266645r^{10}+508715r^{12}-52765r^{14})\sin\theta \\
&-\frac{G^2(1-r^2)\,r^3}{2100}(134-811r^2+785r^4-195r^6)\sin 3\theta \\
&-\frac{G^4(1-r^2)\,r^3}{677376000}(48791-122464r^2+138440r^4-85000r^6 \\
&+28400r^8-4570r^{10}+330r^{12})\sin 3\theta. \qquad (60)
\end{aligned}
$$

4. Discussion

Equations (26), (34), (42) and (60) give the first four terms of the perturbation solution (22) for the axial velocity, and (30), (38) and (52) give the first three terms of the perturbation soluton (23) for the stream function of the secondary flow. Together these give a third-order solution for the flow in a straight pipe that is rotating about an axis oriented at an angle α relative to the pipe axis. The governing equations (16) and (17) had been previously solved for related problems by Barua (1954) and Ito & Motai (1974) using similar methods. Barua obtained a partial second-order solution for the flow in a straight pipe rotating about an axis perpendicular to the pipe axis. The present solution shows that the flows are driven by the component of the rotation that is perpendicular to the axis of rotation and that Barua's solution may be extended to this problem by simply multiplying his rotation parameter by the sine of the non-alignment angle. The first-order and the parts of the second-order solution that he developed are the same as the similar parts in the present solution if $\alpha = 90°$. Ito & Motai (1974) reported a second-order solution for flow in a rotating helically coiled tube of negligible pitch. A limiting case of their analysis, that of infinite helix radius or zero curvature, defaults to Barua's (1954) problem. The terms in their solution for this limiting case agree with the analogous terms in the present problem. (Ito & Motai (1974) give their results in terms of a parameter that is equal to 192 times N_α with $\alpha = 90°$, and Barua (1954) used a parameter equal to 96 times N_α with $\alpha = 90°$.)

The flow rate through the pipe may be obtained by integrating w over the cross-section. Since the first- and third-order terms of the solution for w are odd functions of θ, the flow rate is affected by only the zeroth- and second-order terms. Using (22), (26) and (42) in the integration yields

$$Q_R = \frac{\pi \nu a G}{2}\left[1-\left(\frac{G^2}{448}+4\right)N_\alpha^2\right], \tag{61}$$

in which Q_R represents the flow rate in the rotating non-aligned straight pipe. Since the flow rate Q_S through a stationary straight pipe is $\frac{1}{2}\pi\nu a G$, the ratio of the rate of discharge in a rotating non-aligned pipe to that in a stationary pipe for a specified pressure gradient is

$$\frac{Q_R}{Q_S} = 1-\left[\frac{G^2}{448}+4\right]N_\alpha^2. \tag{62}$$

Equation (61) may be transformed into

$$N_R = G\left[1-\left(\frac{G^2}{448}+4\right)N_\alpha^2\right] \tag{63}$$

by defining $N_R \equiv$ Reynolds number $= 2aW/\nu$, in which W is the cross-sectional average velocity $Q_R/\pi a^2$. An expression for the dimensionless axial pressure gradient G in terms of N_R and N_α may be found by inverting (63) and ignoring higher-order terms:

$$G = N_R\left[1+\frac{(N_R N_\alpha)^2}{448}+4N_\alpha^2\right]. \tag{64}$$

The third-order solution, (22) and (23) along with (26), (30), (34), (38), (42), (52), (60) and (64), may be conveniently written as

$$w = N_R\{\hat{w}_0(r) + [N_R N_\alpha]\,\hat{w}_1(r,\theta) + [N_R N_\alpha]^2\,[\hat{w}_{2a}(r,\theta) + \tfrac{1}{448}\hat{w}_0(r)] + [N_\alpha^2]\,[\hat{w}_{2b}(r,\theta) + 4\hat{w}_0(r)] + [N_R N_\alpha]^3\,[\hat{w}_{3a}(r,\theta) + \tfrac{1}{224}\hat{w}_1(r,\theta)] + [N_R N_\alpha]\,[N_\alpha^2]\,[\hat{w}_{3b}(r,\theta) + 8\hat{w}_1(r,\theta)]\} \quad (65)$$

and

$$f = [N_R N_\alpha]\hat{f}_1(r,\theta) + [N_R N_\alpha]^2\hat{f}_2(r,\theta) + [N_R N_\alpha]^3\,[\hat{f}_{3a}(r,\theta) + \tfrac{1}{448}\hat{f}_1(r,\theta)] + [N_R N_\alpha]\,[N_\alpha^2]\,[\hat{f}_{3b}(r,\theta) + 4\hat{f}_1(r,\theta)], \quad (66)$$

in which parameter-free expansion coefficients $\hat{w}_n$ and $\hat{f}_n$ are introduced by factoring out the parameter from w_n and f_n, i.e. letting $\hat{w}_n = w_n/G^{n+1}$ and $\hat{f}_n = f_n/G^n$ and using (64) to express G in terms of N_R and N_α. Thus $\hat{w}_n$ and $\hat{f}_n$ depend on r and θ only and are independent of the flow regime, i.e the relative size of each term in these series depends on the magnitudes of the parameters but the shape of each term is always the same. The $\hat{w}_0$, $\hat{w}_1$ and $\hat{f}_1$ terms that appear in the second- and third-order terms of these series expansions arise as a result of the higher-order terms in the expression for G in terms N_R and N_α. These equations may be written more compactly by combining the functions of r and θ in each term and introducing the tilde $\sim$ to denote the expansion coefficients in this final form (e.g. $\tilde{w}_0 = \hat{w}_0$ but $\tilde{w}_{2b} = \hat{w}_{2b} + 4\hat{w}_0$):

$$w = N_R\{\tilde{w}_0(r) + [N_R N_\alpha]\,\tilde{w}_1(r,\theta) + [N_R N_\alpha]^2\,\tilde{w}_{2a}(r,\theta) + [N_\alpha^2]\,\tilde{w}_{2b}(r,\theta) + [N_R N_\alpha]^3\,\tilde{w}_{3a}(r,\theta) + [N_R N_\alpha]\,[N_\alpha^2]\,\tilde{w}_{3b}(r,\theta)\} \quad (67)$$

and

$$f = [N_R N_\alpha]\tilde{f}_1(r,\theta) + [N_R N_\alpha]^2\tilde{f}_2(r,\theta) + [N_R N_\alpha]^3 f_{3a}(r,\theta) + [N_R N_\alpha]\,[N_\alpha^2]\tilde{f}_{3b}(r,\theta), \quad (68)$$

in which the functions subscripted a in the second-and third-order terms are the coefficients of $[N_R N_\alpha]^n$, whereas the functions subscripted b are the coefficients of $[N_R N_\alpha]^{n-2} N_\alpha^2$. (Higher-order approximations would produce additional types of terms.) Because they each have a- and b-parts that are multiplied by different combinations of parameters, the second- and third-order terms can have different shapes depending on the magnitudes of the parameters.

These equations indicate that the velocity fields in a rotating non-aligned straight pipe can be expressed as functions of two independent dimensionless parameters: $N_R N_\alpha$ and N_α^2. The first of these, $N_R N_\alpha$, is analogous to the Dean number that characterizes flow through helically coiled tubes with loosely wound coils. It represents the ratio of the product of the inertial and Coriolis forces to the square of the viscous forces. The second parameter, N_α^2, represents the square of the ratio of the Coriolis to viscous forces and corresponds to the coil-ratio parameter that is important in the helical-tube problem with tightly wound coils. The flow that arises in a rotating non-aligned straight pipe, in fact, is qualitatively similar to the flow in a coiled tube. The net flow is a pair of symmetrical screw movements in the two halves of the pipe that are divided by the diameter along $\theta = 90°$, $270°$.

The infinite series could be viewed alternatively as having three types of terms: those that depend on $N_R N_\alpha$, those that depend on N_α^2, and those that depend on products of $N_R N_\alpha$ and N_α^2. If N_α^2 is small but $N_R N_\alpha$ is not, the solution would be a series in powers of $N_R N_\alpha$. If $N_R N_\alpha$ is small but N_α^2 is not, the solution for w would

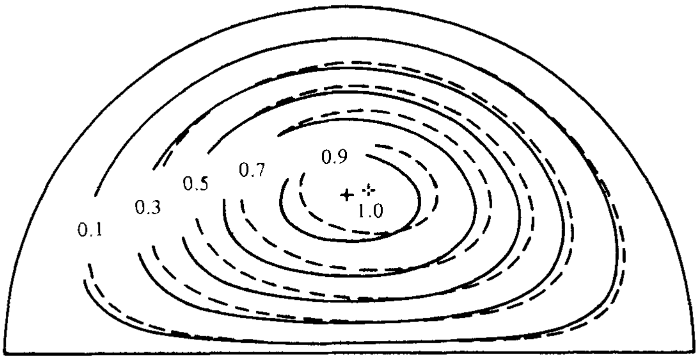

FIGURE 2. Normalized third-order secondary-flow streamlines for two sets of parameter values. The solid lines are for $N_R N_\alpha = 5 \times 10^{-3}$ and $N_\alpha^2 = 2.5 \times 10^{-5}$. The axial flow profile for this case is essentially parabolic. The dashed lines are for $N_R N_\alpha = 3$ and $N_\alpha = 0.04$. The axial flow profile for this case has double maxima located near the centres of the circulations.

be a series in powers of N_α^2. If both parameters are significant, the solution would be a double series expansion in $N_R N_\alpha$ and N_α^2. Mansour (1983) recently considered fully developed steady laminar flow through a pipe that is rotating slowly (small N_α) about a line perpendicular to the pipe axis. He has developed a series solution in powers of $N_R N_\alpha$ and, using computer methods, extended the series to 34 terms. Further, he has shown that the series converges for $N_R N_\alpha < 4.3$. A similar extended series in powers of N_α^2 is not known, nor is an extended double series in powers and products of $N_R N_\alpha$ and N_α^2. Thus the range of validity of the present third-order solution is difficult to estimate. If N_α^2 is restricted to values less than 0.04, however, the magnitude of the third-order term is always less than the magnitude of the first- and second-order terms. Details of the present solution are therefore considered for the range $N_R N_\alpha < 4$ and $N_\alpha^2 < 0.04$.

The qualitative character of the transverse circulations is essentially independent of the parameter values within this range of parameters. Based on considering peak values, the second-order term, the $3a$-term and the $3b$-term in the series for f are always less than 8%, 12% and 3% respectively of the first-order term. Thus the circulation pattern is fundamentally determined by the first-order term with small modifications due to the higher-order terms. Figure 2 shows normalized third-order secondary-flow streamlines for two sets of values of $N_R N_\alpha$ and N_α^2. The solid curves are the streamlines with $N_R N_\alpha = 5 \times 10^{-3}$ and $N_\alpha^2 = 2.5 \times 10^{-5}$, and the dashed curves are the streamlines with $N_R N_\alpha = 3$ and $N_\alpha^2 = 4 \times 10^{-2}$. In the former case the centres of the circulations lie about $r = 0.45$ and very close to $\theta = 0$ and 180° and this is the pattern due to the first-order term. In the latter case the circulation centres are slightly off the $\theta = 0$, 180° diameter, i.e. more toward the outside of the rotation, and are a little more away from the pipe centreline than the former case. The shift is due to minor modifications from the higher-order terms, and results in faster secondary velocities in the vicinity of the wall and slower secondary velocities in the central core. Nevertheless, the qualitative character of the transverse flow is relatively unchanged. The relative strength of the secondary velocities will increase with $N_R N_\alpha$ and N_α^2, but the centres of the circulations change little within the range of parameter values considered.

The axial-velocity profile is also altered by the rotation, and the qualitative character of the $\tilde{w}_n$ terms is shown in figure 3. Equations (34), (42) and (60) and figure 3 indicate that the odd-order terms are odd functions of θ and the even-order terms

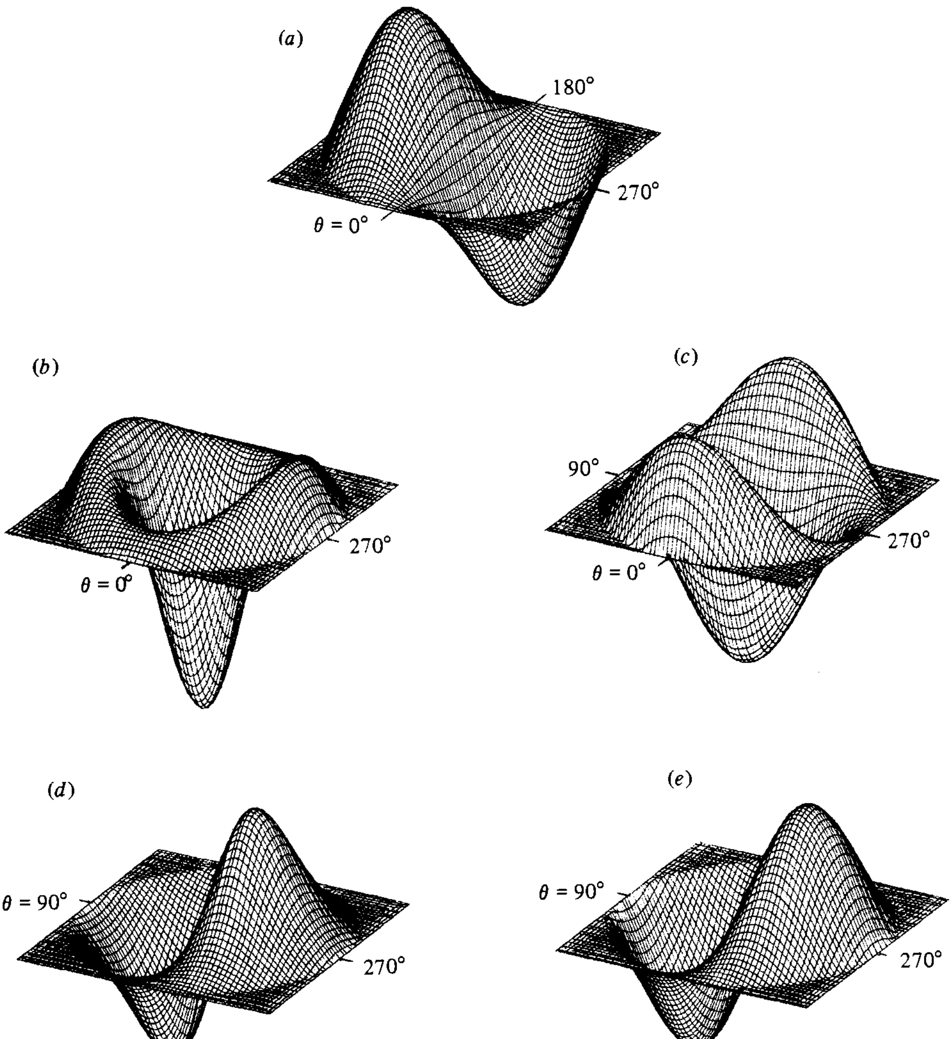

FIGURE 3. The qualitative shape of the terms $\tilde{w}_n$ in the series expansion for the axial velocity: (*a*) the $\tilde{w}_1$ term; (*b*) the $\tilde{w}_{2a}$ term; (*c*) the $\tilde{w}_{2b}$ term; (*d*) the $\tilde{w}_{3a}$ term; (*e*) the $\tilde{w}_{3b}$ term. The peak-to-peak magnitudes of these terms, relative to a $\tilde{w}_0$ term of unity, are 7.28×10^{-2}, 5.55×10^{-3}, 6.75, 1.08×10^{-3} and 1.45 for $\tilde{w}_1$, $\tilde{w}_{sa}$, $\tilde{w}_{2b}$, $\tilde{w}_{3a}$ and $\tilde{w}_{3b}$ respectively.

are even functions of θ. Each term is normalized to illustrate their shape better. The actual peak-to-peak magnitude of the terms, relative to a $\tilde{w}_0$ term of unity, is 7.28×10^{-2}, 5.55×10^{-3}, 6.75, 1.08×10^{-3} and 1.45 for $\tilde{w}_1$, $\tilde{w}_{2a}$, $\tilde{w}_{2b}$, $\tilde{w}_{3a}$ and $\tilde{w}_{3b}$ respectively. The effect of each term in modifying the basic parabolic profile depends, of course, not ony on these magnitudes but also on the size of the parameters $N_R N_\alpha$ and N_α^2 for the flow. The first-order term $N_R N_\alpha \tilde{w}_1$ skews the parabolic profile toward the outside of the rotation along $\theta = 90°$. The second-order terms $[N_R N_\alpha]^2 \tilde{w}_{2a}$ and $[N_\alpha^2] \tilde{w}_{2b}$ reduce the peak centreline velocity of the parabolic profile and increase the velocities near the pipe walls. The a-term boosts the near-wall velocities in an approximately axisymmetric manner. The b-term distorts the single axisymmetric peak of purely parabolic flow into a diametrically symmetric double maximum with the peaks occurring along $\theta = 0°$ and 180°, i.e. toward the sides of the tube. The

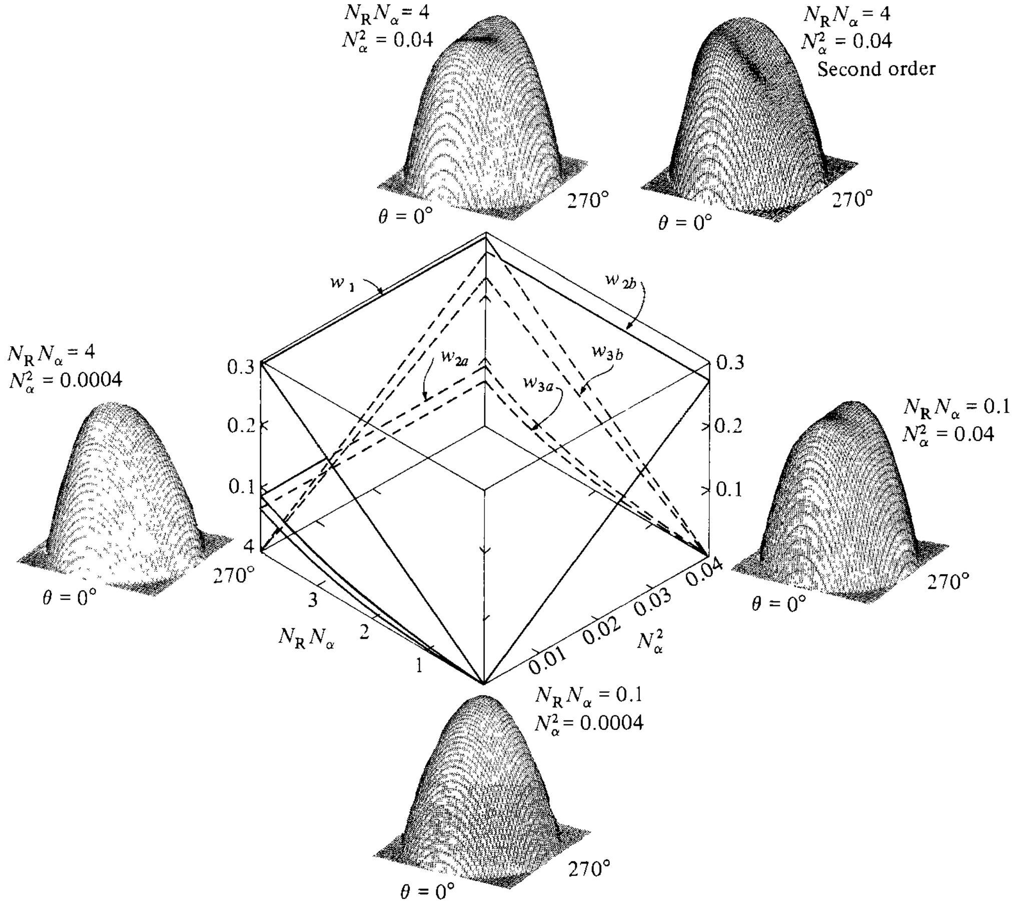

FIGURE 4. The centre diagram shows the peak-to-peak sizes, relative to a zero-order term of unity, of the first-, second- and third-order terms for the axial velocity as functions of $N_R N_\alpha$ and N_α^2. Note that the a-terms, i.e. the parts of the second- and third-order terms that are powers of $N_R N_\alpha$, are always small relative to the w_1 term in this range of $N_R N_\alpha$ and N_α. The figures in the corners are the third-order axial-velocity profiles for the indicated values of the parameters. The extra figure to the right of the third-order profile with $N_R N_\alpha = 4$ and $N_\alpha^2 = 0.04$ is the profile calculated using only second-order theory. The third-order terms have the effect of negating the skewing to the outside that is produced by the first-order term.

third-order terms $[N_R N_\alpha]^3 \tilde{w}_{3a}$ and $[N_R N_\alpha][N_\alpha^2]\tilde{w}_{3b}$ are qualitatively similar to the first-order term except that they are of opposite signs. They tend to negate the effects of $\tilde{w}_1$ and skew the profile toward the inside of the rotation.

The relative size of each term, as a function of $N_R N_\alpha$ and N_α^2, is shown in figure 4. The w_1, w_{2a} and w_{3a} terms are functions of $N_R N_\alpha$ only, w_{2b} is a function of N_α^2 only, and w_{3b} is a function of both parameters. Note that, within this range of parameters, the a-terms are always small. Both w_{2b} and w_{3b}, however, can be quite significant. If N_α^2 is small but $N_R N_\alpha$ is significant (e.g. flow with slow rotation or a small angle of non-alignment), the axial profile is determined by the sum of the w_0, w_1, w_{2a} and w_{3a} contributions. The w_{2a} and w_{3a} terms, however, are small compared with the w_1 term, and the axial profile is essentially a parabola skewed toward the outside of the rotation. If $N_R N_\alpha$ is small but N_α^2 is significant (e.g. a low flow rate in a tube with rapid rotation and large angle of non-alignment) and axial profile is determined by w_0 and w_{2b} and has a double maximum along the diameter on $\theta = 0°$ and 180°.

In this case enough slow-moving wall-fluid is brought to the centreline by the transverse circulations that the profile has a minimum rather than a maximum near the centreline. If both $N_R N_\alpha$ and N_α^2 are of significant size the axial profile will be skewed and have two peaks, one on each side of the $\theta = 90°$, $270°$ diameter. Figure 4 shows the third-order axial profiles for these cases. Also shown on figure 4 is a second-order axial profile for the case with $N_R N_\alpha = 4$ and $N_\alpha^2 = 0.04$. The effect of the third-order correction is to remove the outward skewing effect of the first-order term and, in fact, to produce a slight skewing toward the inside of the rotation.

These changes in axial profiles can be significant in the performance of heat/mass-transfer devices with secondary circulations. Without transverse secondary motions the heat or mass transferred to the fluid is concentrated near the walls, while the axial convection, and its parabolic profile, is heavily weighted to fluid elements near the centreline. Induced transverse circulations enhance the transfer efficiency by redistributing some of the heat/mass from near the wall to the high-velocity regions near the centre. As shown here, however, the conditions for inducing strong secondary flows can cause changes in the axial profile such that the high velocities, under some conditions, are relocated to regions of low temperature/concentration in the redistributed heat/mass field. If the conditions are such that the axial profile remains essentially parabolic the redistribution of heat/mass greatly improves the axial transport. In some flow regimes, on the other hand, increases in the magnitude of the parameters, which increase the strength of the circulations, can actually decrease the transfer efficiency by disadvantageously altering the axial-transport weighting function, i.e. the axial-velocity profile. For example, the region of low temperature/concentration along the diameter through $\theta = 90$, $270°$ is moved by the secondary circulations from the pipe centreline to a position between the centerline and the wall. If the parameter values are such that w_1 is significant, e.g. $N_R N_\alpha$ large, N_α^2 small, the axial profile is skewed outward along this diameter and the region of highest velocity moves from the centreline to the region of lower concentration between the centreline and the wall. Further, with transverse circulations the regions of lowest temperature/concentration are located near the circulation centres in each half-circle area (see figure 2) (with weak transverse convection, the transfer process is by conduction/diffusion only in these regions). As mentioned above, the location of these circulation centres (i.e. the regions of lowest temperature/concentration) is insensitive to the parameter values. If N_α^2 is relatively large the axial profile is altered (see figure 4) such that the maximum axial velocities are located near these regions of lowest temperature/concentration. The enhanced transport effect of the transverse secondary velocities can be somewhat counteracted, again, by this redistribution of axial velocities, since the regions of low temperature/concentration are emphasized in the axial convection.

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